MECHANICAL MEASUREMENTS

PECULIARITIES OF THE RESPONSE OF A LASER INTERFERENCE
GRAVITATIONAL ANTENNA TO LOW-FREQUENCY DISTURBANCES

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The extensive use of Fabry-Perot interferometers for measuring small relative displacements in applied [1] and fundamental [2] (for instance, recording of gravitational waves [3]) research calls for study of the possible mechanisms of transformation of low-frequency noise disturbances to high-frequency types of approximately the same frequency as the corded useful signal. The study acquires considerable interest in view of the development of wide-base laser interference gravitational antennas [4]. At the same time, the high cost, large overall dimensions, and increased complications in manufacture and operation of such installations make it necessary, in the first instance, to analytically define the transformation of low-frequency seismic disturbances to the range of working frequencies (-1 kHz) and, based on this, to optimize laser interference gravitational antennas for reducing the level of random measuring errors.

Laser interference gravitational antennas being developed now are interferometers built up on Michelson's circuit, each arm of which is assembled as a Fabry-Perot resonator. The resonator mirrors \( S_1 \) and \( S_2 \) (\( S_1 \) is a partially transmitting mirror) are mounted on the test bodies which are weakly connected with the base. Since a Fabry-Perot resonator is a multi-beam interferometer, then, radiation pressure on mirror \( S_2 \) (and, consequently, on the test body) depends significantly on its possible displacement in the direction of the resonator's optical axis. Such displacements can occur as a result of low-frequency microseismic [5] and a series of other disturbances (thermal noise, geomagnetic pulsations, effect of cosmic rays, et al. [6]). It is of interest to examine the possibility of transformation of these low-frequency disturbances to high-frequency fluctuations of the signal recorded by the interferometer. This transformation can be caused, in particular, by nonlinearity of the equation of motion of the test body in the radiation pressure field.

We examine a Fabry-Perot optical resonator, the radiation source for which is a laser of intensity \( I_0 \), creating amplitude \( E_0 \) of the light wave at the input to the resonator. We assume that mirror \( S_1 \) is characterized by energy reflection coefficient \( R_1 \) and transmission factor \( \eta_1 \) and, mirror \( S_2 \), by \( R_2 \) and \( \eta_2 \). In this, losses are considered to be small, i.e., \( 1 - (R_i + \eta_i) \ll 1 \), \( i = 1, 2 \). We assume that one of the test bodies with mirror \( S_2 \) mounted on it undergoes an irregular action leading to its displacement from the equilibrium position by amount \( x(t) \). Then, assuming that the direction of propagation of the beam coming from the laser is positive, we can write the amplitude of ray \( n \) incident on the mirror as

\[
E_n^* = E_0 \sqrt{\eta_i^{(n-1)/2}} \cos[\omega_d t + \varphi_n + (n-1)\alpha - k(2L(n-1)+x(t)+\sum_{l=1}^{n-1} x(t-\Delta t)\}] \cos[\omega_d t + \varphi_n],
\]

where \( \omega_c \) is cyclic frequency of laser radiation; \( \varphi_n \) is initial phase; \( \alpha \) is total losses of the phase with one-time rererection from the resonator mirrors; \( k \) is wave number; \( L \) is undisturbed length of resonator; \( \Delta t = 2L/c \) is time interval between successive rerefections of the ray by mirror \( S_2 \); \( c \) is velocity of light. Amplitudes for the rays reflected from mirror \( S_2 \) and emergent from the resonator are equal to

\[
E_n' = V\eta_i R_2 E_n, \quad E_n'' = V\eta_i R_2 F_n E_n.
\]

The following expressions will respectively characterize the intensity of radiation pressure \( F(x, t) \) on mirror \( S_2 \) and intensity of radiation \( I(x, t) \) emergent from the resonator:

\[
F(x, t) = (2L/e^2)\delta(1-R_2) \langle E^2 \rangle; \quad I(x, t) = (2L/e^2)\delta F_n \langle E^2 \rangle.
\]


where \( \langle E^a \rangle = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=1}^{N} E_n^a \right)^{\frac{1}{2}} \) is the averaging operation for one period of the light wave. Considering that average amplitude of oscillations of the test body \( \langle A_x \rangle \) is small, i.e., \( \langle A_x \rangle \ll L/N_0 \) (\( N_0 \) is average number of rerefections in the resonator), and assuming that relationship \( 2kL - \alpha = 2\pi n + \kappa \) (\( n \) is a whole number; \( \kappa \) is phase shift characterizing resonator tuning) takes place for the resonator, we can obtain the following expression:

\[
\langle E^a \rangle = (E_0^a/2)^{\frac{1}{2}} \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=1}^{N} (R_1 R_4)^{n-1} + 2 \sum_{n=m}^{\infty} (R_1 R_4)^{(n+m-1)} \left( \sum_{k=0}^{\infty} x(t - \kappa \Delta t) + (n-m)x \right) \right\}.
\]

(2)

If in case oscillations of the test body are low-frequency type, and their limiting frequency \( \nu = 1/N_0 \Delta t \), then \( x(t - \kappa \Delta t) \) in (2) can be replaced by \( x(t) \). Accounting for the fact that losses in one cycle of rerefections of the ray in the resonator \( \Delta = 1 - \sqrt{R_1 R_2} \) are small (\( \Delta \ll 1 \)), expression (2) takes the following simple form after summation:

\[
\langle E^a \rangle = (E_0^a/2)/(\Delta^2 + 4(1 - \Delta)\sin^2(kx(t) + \kappa/2)).
\]

(3)

Substitution of (3) in (1) yields the final expressions for the intensity of radiation pressure on the test body \( P(x) \) and intensity of radiation emergent from the resonator \( I(x) \).

If the resonator is out of adjustment such that \( \Delta < kx + \kappa/2 < 1 \), and this corresponds to the working point for Fabry–Perot interferometers, then the expression for the intensity of radiation pressure on the test body can be represented as

\[
P(x) = (I_0^a/2k)(x + \kappa)^2,
\]

(4)

where the notation \( r = x/2k \) is introduced and we take into account that \( 1 - R_2 \ll 1 \).

Substitution in (4) of the parameters \( I_0 = 10^2 \text{ W} \), \( k = 10^7 \text{ m}^{-1} \), \( \kappa = 0.01 \), and \( n_1 = 10^{-3} \), characteristic for installations under construction in the California and Massachusetts Institutes of Technology [4], yields \( P = 10^{-8} \text{ to } 10^{-6} \text{ N} \). Taking into account its nonlinear dependence on \( x \), such an intensity can substantially change the character of motion of the test body.

The equation of oscillations of the test body in the radiation pressure field can be written as

\[
x + 2\beta\dot{x} + \omega_0^2 x - b/(x + \kappa)^2 = \xi(t),
\]

(5)

where \( \beta \) is damping coefficient; \( \omega_0 \) is the natural frequency of oscillations of test body in the absence of radiation pressure; \( b = I_0 n_1/2Me^2 \); \( \xi(t) \) is coercing force; \( M \) is test body mass. If \( \xi(t) \) can be approximated by white Gaussian noise with intensity 2D (this can, for example, be microseismic noise), then, the Fokker–Planck equation [7] can be constructed for (5).
This equation gives the distribution function \( f(x, p, t) \) of fluctuations of coordinate \( x \) and pulse \( p = Mx \) of the test body as follows:

\[
\frac{\partial f}{\partial t} + \left( p/M \right) \frac{\partial f}{\partial x} + \left[ M \dot{b}((x+r)^2 - 2b) - Mx^2 \right] \frac{\partial f}{\partial p} - 2b \frac{\partial f}{\partial p} = \cdot \frac{\partial f}{\partial p} .
\]  

(6)

A stable solution of \( f(x, p) \) is the combined probability density of coordinate \( x \) and pulse \( p \) of the test body. Integration of the distribution function \( f(x, p) \) for all possible values of the pulse makes it possible to obtain the expression for the steady distribution of fluctuations of test body with respect to coordinate \( x \):

\[
f_0(x) = \exp \left[ - \left( M \omega_0^2 x^2 / 2 - V(x) \right) / W \right].
\]

(7)

where

\[ S^{-1} = \int \exp \left[ - \left( M \omega_0^2 x^2 / 2 - V(x) \right) / W \right]; \]

\( V(x) = -M \dot{b} / (x + r) \) is potential energy of radiation pressure field; \( W = 28M/D \) is average energy or random oscillations of test body.

The distribution function (7) differs from the Gaussian. This is connected with the nonlinear character of oscillations described by (5) and points to the possibility of occurrence of the high-frequency components of fluctuations of the coordinate of the test body. Graphs of \( f_0(x) \) calculated by (7) for an interference gravitational antenna with \( M = 10^2 \) kg [8] and \( \omega_0^2 = 20 \) sec\(^{-2} \) having the following values of energy \( W \) of random vibrations of the test body and coefficient \( b \) (characterizing the parameters of the laser-resonator system) are given in Fig. 1: 1) \( W = 10^{-15} \) J and \( b = 10^{-27} \) J·m/kg; 2) \( W = 10^{-16} \) J and \( b = 10^{-26} \) J·m/kg; 3) \( W = 10^{-15} \) J and \( b = 10^{-26} \) J·m/kg; 4) \( W = 10^{-14} \) J and \( b = 10^{-26} \) J·m/kg; 5) \( W = 10^{-15} \) J and \( b = 10^{-25} \) J·m/kg. From comparison of the graphs, it follows that with increase in radiation pressure, characterized by parameter \( b \), the equilibrium position of the test body is displaced (graphs 1, 3, 5). This leads to variation in the operating point of the resonator. Increase in energy \( W \) causes broadening of the distribution function \( f_0(x) \) and intensification of distortions of its form in comparison with a Gaussian distribution (graphs 2, 3, and 4).

We make use of the following method to determine the character of transformation of low-frequency oscillations to high-frequency oscillations of its fluctuations. We assume that after a single external pulse action, caused, for instance, by the passage of a seismic wave, the test body has acquired energy \( W \) and the dissipation of this energy in time \( \tau = 1/\beta \), during which, the oscillations can be assumed to be free. Assuming that the oscillating system is of adequately high quality \( \beta \ll \omega_0 \), its period corresponding to energy \( W \) can be determined by the formula [9].

\[
\tau(W) = \frac{1}{\beta} \sqrt{\frac{2Mdx}{W - \Pi_\theta(x)}} .
\]

where \( \Pi_\theta(x) = M\omega_0^2 x^2 / 2 - V(x) \), is total potential energy of the test body in the gravitational force and radiation pressure fields; \( x_1 \) and \( x_2 \) are values of the coordinates at which the potential energy \( \Pi_\theta(x) \) becomes equal to \( W \), i.e., \( \Pi_\theta(x_{1,2}) = W \).
The period of oscillations of the test body as applied to the examined system, takes the form

\[ T(W) = \frac{2}{\omega_0} \sum_{i=1}^{n} \sqrt{\frac{1}{r_i^2} + \frac{1}{(x_i^2 - r_i^2)^2} + \frac{1}{(x_i^2 + r_i^2)^2}}. \]  

(8)

where the following notations are introduced: \( a_1 = \frac{2W}{Mw_0^2} \); \( a_2 = \frac{2(Mb - Wr)}{Mw_0^2} \).

Finding the integral in (8) gives

\[ T(W) = 2(x_1 + r)\omega_0 \sqrt{(x_1 + r)(x_3 - x_1)} \prod_{i=1}^{2} (\pi/2, \rho, q) \]

(9)

where \( x_1, x_2, \) and \( x_3 \) are roots of the cubic equation in the denominator of integral (8) and, here, \( x_1 > x_2 > x_3; \prod_{i=1}^{2} (\pi/2, \rho, q) \) is the elliptical integral of third order; \( \rho = (x_2 - x_1)/(x_1 + r); q = \sqrt{(x_2 - x_1)(x_3 + r)}/(x_1 + r)(x_2 - x_1). \)

Substitution of the values characteristic of the laser-resonator system \( (W = 10^{-14} \text{ J}, b = 10^{-26} \text{ J.m/kg}, \omega_0 = 4.5 \text{ sec}^{-1}, \) and \( r = 5 \cdot 10^{-10} \text{ m} \) in (9) makes it possible to determine the natural oscillations and, hence, even their frequency \( \omega = 2\pi/T(W) = 9.6 \text{ sec}^{-1} \) which is two times higher than frequency \( \omega_0 \).

The free oscillations of the test body in time \( \tau < 1/\beta \) after passage of the seismic wave can be described by equation (5) for \( \xi(t) = 0 \) and in disregarding the term \( 2\beta x \) which accounts for damping. The periodic solution for \( x(t) \) of the equation thus obtained is presented in [9] as a Fourier series for the natural frequency \( \omega \) as follows:

\[ x(t) = A_0 + \sum_{i=1}^{\infty} A_i \cos(\omega t). \]  

(10)

Substitution of (10) in the examined equation of free undamped oscillations of the test body gives

\[ (A_0 + r)\omega_0^2 A_n - \sum_{i=1}^{\infty} \alpha_i \Delta_i \cos(\omega_0 t) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta_{ij} \Delta_i \Delta_j \cos(\omega_0 t) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \gamma_{ij} \Delta_i \Delta_j + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{ij} \Delta_i \Delta_j = 0, \]

(11)

where

\[ \alpha_i = (A_0 + r)^2 \gamma_i + 2(A_0 + r) \omega_0 A_i; \beta_{ij} = 2(A_0 + r) \gamma_i + \alpha_i^2 \gamma_i; \]

\[ \gamma_i = \epsilon_i^2 - k^2 \omega_0^2. \]

Considering that \( A_1 < A_0 + r, A_{i+1} > A_1, \) and \( i = 1, ..., n \), the relationship (11) makes it possible to determine the recurrence formula for amplitude \( A_i \) higher than the first order after grouping terms having the same frequency \( \omega_i = \epsilon_i \omega \) and accounting for triviality of terms containing the amplitude \( A_i \) for \( i > 1 \), i.e.,

\[ A_i = -\frac{1}{2} \sum_{j=1}^{i-1} \beta_{jk} A_k A_{i-k} - \frac{1}{4} \sum_{j=1}^{i-1} \Delta_i \Delta_j A_j A_{i-j-1} A_{i-j-1} + 2 \gamma_i A_k A_{i-k} - \frac{1}{4} \sum_{j=1}^{i+k-1} \Delta_i A_i \Delta_j A_{i-j}, \quad i = 1, ..., n. \]

(12)

where \( \beta_{jk} = \beta_{jk} / \alpha_k, \gamma_k, \gamma_i = \gamma_k / \alpha_k, \) \( \Delta_i \Delta_j = \sum_{i=1}^{i+k-1} \) denotes summation of all terms with \( i \geq 1 \) and \( k \geq 1 \) under the condition that \( i + k = i \).

Values of \( A_0 \) and \( A_1 \) depend on \( W \) and \( b \) are required to be set for direct calculation of amplitude \( A_k \) by (12). The amount of steady displacement can be estimated in the first approximation from the position of the maximum in the graphs in Fig. 1 while the amplitude of the first harmonic \( A_1 \) is connected with energy \( W \) by the approximate relationship \( W = M b/(A_0 + r) + M w_0 A_0^2/2 \).

We analyze the dependence of the amplitude of oscillations on frequency \( |A_{i}(\omega_k)| \) after setting values of \( b = 10^{-26} \text{ J.m/kg}, W = 10^{-15} \cdot 10^{-16} \text{ J}, \) and \( r = 5 \cdot 10^{-10} \text{ m} \) and \( A_1 = 5 \cdot 10^{-10} \cdot 10^{-19} \text{ m} \), and considering that the other parameters of the laser-resonator system are approximately the same as for the gravitational antennas under construction.

Graphs of the dependence of \( |A_{i}(\omega_k)| \) with the following values of \( A_i \) are given in Fig. 2 in logarithmic scale: 1) \( A_1 = 0.5(A_0 + r); 2) A_1 = 0.9(A_0 + r); 3) A_1 = 0.95(A_0 + r). \) As
it follows from the graphs, the amplitude of the hundredth harmonic \( A_{100} \) for the case \( A_1 = 0.9 \ (a_0 + r) \) is in the region of \( 10^{-15} \) m, while for \( A_1 = 0.95 \ (a_0 + r) \) it takes a value of approximately \( 10^{-13} \) m. With growth in energy of random low-frequency oscillations of the test body, there is corresponding increase in amplitude of the high-frequency harmonics and it can exceed the proposed level of sensitivity \( (\Delta x = 2 \cdot 10^{-15} \) m) in the working frequency range of about 1 kHz for gravitational antennas under construction. Variations in the metric of \( h = 10^{-20} \) with interferometer base \( L = 4 \cdot 10^3 \) m correspond to the sensitivity level \( \Delta x = 2 \cdot 10^{-15} \) m.

The results of a series of works in which experiments on the study of the effect of various noises [10], among them seismic [5], are discussed, enable us to draw a conclusion on the reality of the indicated values of energy of low-frequency random oscillations of a test body.

Thus, the examined mechanism of the transformation of low-frequency oscillations of a test body to its high-frequency fluctuations can significantly limit the sensitivity of laser interference gravitational antennas. In this case, as it follows from (1) and (3), the proposed mechanism connected with radiation pressure on the test body leads to irregular high-frequency fluctuations of the intensity of light radiation I emergent from a Fabry-Perot resonator and, consequently, to increase in the high-frequency noise level of the gravitational antenna. In order to reduce the effect of this mechanism of high-frequency random phenomenon, it is necessary during design of these antennas to either reduce the laser light source intensity \( I_0 \), or increase the parameter \( a \), by shifting the operating point of the resonator to a lower sensitivity region.

LITERATURE CITED

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