Use of the theory of non-Markovian processes in the description of Brownian motion

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A method of statistical description of Brownian motion in a physical medium with fluctuating transport coefficients is proposed. Equations are obtained for the n-dimensional characteristic functions that describe the momentum fluctuations of a Brownian particle, and it is shown that in the first approximation the fluctuations of the friction coefficient can be taken into account by constructing and solving an equation for a two-dimensional distribution function.
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1. INTRODUCTION

The problem of creating an adequate theory of Brownian motion is directly related to the problem of constructing a description that takes into account the statistical characteristics of the real physical processes acting on a Brownian particle. The causes of anomalous behavior in Brownian motion may include regular and irregular inhomogeneities of the physical properties of the medium,[1] large-scale correlations of the random effects acting on the Brownian particle,[2] and nonequilibrium fluctuations in the physical parameters of the medium.[3] In all these cases, the additional effects can be taken into account by constructing a description of Brownian motion in a medium with physical properties that change in accordance with *a priori* known laws.[4] The necessary calculations involve the use of numerical and analytical methods and are usually restricted to the solution of a specific problem.

Somewhat different in nature is the problem of constructing a theory of Brownian motion that takes into account a bounded number of particles of the medium that interact simultaneously with a Brownian particle.[5,6] The interaction of the Brownian particle with only some, and not all, of the particles of the medium that at a given instant of time are in direct contact with it makes it necessary to take into account fluctuations of the transport coefficients in the construction of the theory of the Brownian motion.[7] This is because the formation of the transport coefficients has a statistical nature, and the values of these coefficients depend on the nature of the interaction of the Brownian particle with the particles of the medium. The fluctuations of the transport coefficients must be most strongly manifested when only a small number of particles of the medium interact simultaneously with the Brownian particle.

This physical effect has fundamental importance, and it does not appear possible to take it into account if one remains in the framework of the Markov theory of Brownian motion. It is necessary to construct adequate methods of description that take into account the fluctuations of the transport coefficients that actually exist in the physical medium and the manner in which the effect of the particles of the medium on the Brownian particle differs from a Wiener random process. Therefore, the problem arises of developing a non-Markov theory of Brownian motion in which the momentum fluctuations of the Brownian particle are described by means of multidimensional characteristic functions. The form of the equations of the non-Markov theory must depend on the form of the *a priori* known characteristic functions that describe external random effects.

2. THE TRADITIONAL METHOD OF DESCRIBING BROWNIAN MOTION

In the traditional approach, Brownian motion is described by means of a Langevin equation, which in the case of a one-dimensional linear medium has the form[8]

$$\dot{P}+\gamma P=X(t), \tag{1}$$

where P is the momentum of the Brownian particle, γ is the friction coefficient that characterizes the energy dissipation of the Brownian particle as it interacts with the particles of the medium, and $X(t)$ is a random force that describes the effect of the particles of the medium on the Brownian particle. It is assumed that the random force $X(t)$ can be described by a white-noise process with zero mathematical expectation and a Gaussian distribution. The $X(t)$ process is related to a Wiener random process $W(t)$ by[9]

$$W(t)=W(t_0)+\int_{t_0}^{t}X(\tau)d\tau. \tag{2}$$

The friction coefficient γ is determined from the Kubo formula,[10] which in the case of one-dimensional motion of the Brownian particle has the form

$$\gamma=2A\int_0^{\infty}\langle X(0)X(\tau)\rangle d\tau, \tag{3}$$

where we have introduced the notation

$$A=\frac{1}{2mkT}, \tag{4}$$

in which m is the mass of the Brownian particle, k is Boltzmann's constant, and T is the temperature of the medium. The operation $\langle ... \rangle$ denotes averaging over all the variables of the motion of the particles of the medium.

In the under consideration case, the kinetics of the Brownian particle is described by the Fokker–Planck equation[8]

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$$\frac{\partial f}{\partial t}=D\,\frac{\partial}{\partial P}\left(\frac{\partial}{\partial P}+\frac{P}{mkT}\right)f, \tag{5}$$

the solution of which gives the distribution function $f(P;t)$, which completely describes the statistics of the Brownian particle. Here $D=\gamma mkT$ is the diffusion coefficient. Multidimensional distribution functions can be obtained in accordance with the formula[9]

$$f_n(P_1,\dots,P_n;t_1,\dots,t_n)=f(P_n;t_n|P_{n-1};t_{n-1})\dots f \times(P_2;t_2|P_1;t_1)f(P_1;t_1), \tag{6}$$

where the transition distribution functions $f(P_l;t_l|P_{l-1};t_{l-1})$, $l=\overline{2,n}$, are the solution of Eq. (5). Here P_l is the momentum of the Brownian particle at the time t_l.

This method gives a consistent closed description of Brownian motion and is based on the assumption that the effect of the particles of the medium on the Brownian particle can be completely described by a Wiener random process and that the momentum $P(t)$ of the Brownian particle is a Markov random process.

3. THE METHOD OF MULTIDIMENSIONAL DISTRIBUTION FUNCTIONS

A shortcoming of the traditional method of describing Brownian motion is the assumption that the random process $X(t)$ corresponds to Gaussian white noise. However, for a real physical medium it is a consequence of the bounded number of particles of the medium that interact simultaneously with the Brownian particle that this assumption is only a first approximation to the actually existing random process $\widetilde{X}(t)$. Similarly, the assumption that the coefficient of friction γ can be calculated with absolute accuracy by means of the expression (2) is also not sufficiently well founded, since this expression assumes averaging over the variables of the motion of all of the particles of the medium and over an infinite time. In reality, the friction coefficient is formed at each point of the trajectory of the Brownian particle by just those particles of the medium that interact with it at the given time.

The method of multidimensional distribution functions[7,11,12] has been proposed to solve these problems. In it, the Brownian motion is described by solving equations for the multidimensional distribution functions $f_n(P_1,\dots,P_n;t_1,\dots,t_n)$. In particular, for the present case described by the Langevin equation (1) the equation for the two-dimensional distribution function $f_2(P_1,P_2;t_1,t_2)$ has the form

$$\left(\frac{\partial}{\partial t_1}-\hat{T}_1^{(f)}\right)\left(\frac{\partial}{\partial t_2}-\hat{T}_2^{(f)}\right)f_2=\Delta\hat{T}_2^{(f)}f_2, \tag{7}$$

where

$$\hat{T}_l^{(f)}=D\,\frac{\partial}{\partial P_l}\left(\frac{\partial}{\partial P_l}+\frac{P_l}{mkT}\right),\quad l=1,2,$$

$$\Delta\hat{T}_2^{(f)}=2D\,\delta(t_2-t_1)\,\frac{\partial}{\partial P_1}\frac{\partial}{\partial P_2}+2D^2\tau_0\delta(t_2-t_1)\,\frac{\partial^2}{\partial P_1^2}\frac{\partial^2}{\partial P_2^2}+\Delta D(t_1,t_2)\,\frac{\partial}{\partial P_1}\frac{\partial}{\partial P_2}\left(\frac{\partial}{\partial P_1}+\frac{P_1}{mkT}\right)\left(\frac{\partial}{\partial P_2}+\frac{P_2}{mkT}\right). \tag{8}$$

Here $D=\langle D(t)\rangle$ is the averaged diffusion coefficient, τ_0 is the time required for randomization of the particles of the medium, and $\Delta D(t_1,t_2)$ is a coefficient that describes the fluctuations of the friction coefficient:

$$\Delta D(t_1,t_2)=\langle D(t_1)D(t_2)\rangle-\langle D(t_1)\rangle\langle D(t_2)\rangle. \tag{9}$$

The two-dimensional distribution function $f_2(P_1,P_2;t_1,t_2)$ makes it possible to determine the one-dimensional distribution function $f(P;t)$. If it is necessary to find distribution functions of higher order, then equations for the corresponding n-dimensional distribution functions must be derived and solved. These equations can be found by the method presented in Ref. 7.

4. THE STOCHASTIC DIFFERENTIAL EQUATION OF THE NON-MARKOVIAN THEORY OF BROWNIAN MOTION

To construct the non-Markovian description of Brownian motion, we transform Eq. (1) to the form

$$dP=-P\,\widetilde{\gamma}(t)dt+d\widetilde{W}(t), \tag{10}$$

where $\widetilde{\gamma}(t)$ is a random process that must be determined in accordance with the formula

$$\widetilde{\gamma}(t)=2A\int_{-\infty}^{t}\langle\widetilde{X}(t)\widetilde{X}(\tau)\rangle d\tau, \tag{11}$$

and the random process $\widetilde{W}(t)$ is related to the process $\widetilde{X}(t)$ by the expression

$$\widetilde{W}(t)=\widetilde{W}(t_0)+\int_{t_0}^{t}\widetilde{X}(\tau)d\tau. \tag{12}$$

The random process $\widetilde{W}(t)$ is assumed to be different from the Wiener random process $W(t)$. Equation (10) and the integrals (11) and (12) are assumed to be expressed in the Stratonovich form.[9,13]

We represent Eq. (10) in the form

$$dP=-P\,d\widetilde{\theta}(t)+d\widetilde{W}(t), \tag{13}$$

where the increment $d\widetilde{\theta}(t)=\widetilde{\gamma}(t)dt$ of the random process is found in accordance with a formula that can be obtained by means of the integrals (11) and (12):

$$d\widetilde{\theta}(t)=A\langle d\widetilde{W}^2(t)\rangle. \tag{14}$$

If $\widetilde{W}(t)$ is a Wiener random process $W(t)$, the increment $d\widetilde{\theta}(t)$ of the process becomes a deterministic variable $d\theta(t)$ and takes the form

$$d\theta(t)=\gamma dt. \tag{15}$$

The expression (15) is obtained with allowance for the fact that in accordance with Ref. 13

$$\langle dW^2(t)\rangle = \nu dt, \tag{16}$$

where ν is the intensity of the Wiener process $W(t)$.

Equation (13) is an equation in the Stratonovich form. Just as for Eq. (1), the Stratonovich form and the Itô form must be identical for Eq. (13). This is possible only if the random processes $P(t)$ and $d\tilde{\theta}(t)$ in Eq. (13) are independent. This requirement imposes important restrictions on the form of the characteristic function $\tilde{\theta}(t)$ of the process.

Equation (13) can be the same in the Stratonovich form and the Itô form if the additional terms[9] that arise in going from the one representation to the other are of higher order. For the equation (13) in question, this means that the terms containing the factors $(d\tilde{\theta}(t))^2$ must be of order $(dt)^2$. Therefore, when we consider the actual form of the characteristic function $\tilde{\theta}(t)$ of the process we must test whether this condition holds. In particular, this requirement is satisfied by a process $\tilde{\theta}(t)$ with one-dimensional characteristic function of the form

$$h_1(\eta;t) = \exp(i\gamma\eta t). \tag{17}$$

In this case, the one-dimensional distribution function has the form[9]

$$f_1(\theta;t) = \delta(\theta - \gamma t), \tag{18}$$

where $\delta(...)$ is the delta function. It is readily established that the function (18) satisfies all the requirements imposed on the distribution functions, since it is positive definite, vanishes at infinity, and has real positive-definite moments of all orders.

For the further development of the non-Markovian theory of Brownian motion, we represent Eq. (13) in the form

$$dP = b(P)d\tilde{\Psi}, \tag{19}$$

where $b(P) = [-P1]$ is a row vector, and $d\tilde{\Psi} = [{}^{d\tilde{\theta}}_{dW}]$ is a column vector. Equation (19) is the point of departure for constructing the non-Markovian description of Brownian motion with allowance for fluctuations of the coefficient of friction.

5. EQUATIONS FOR THE *n*-DIMENSIONAL CHARACTERISTIC FUNCTIONS

With allowance for the form of Eq. (19), the equation for the one-dimensional characteristic function $g_1(\lambda;t)$ of the random process $P(t)$ has the form[9]

$$\frac{\partial g_1}{\partial t} = \hat{T} g_1, \tag{20}$$

where

$$\hat{T} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\mu [\chi_1(b(P)^T\lambda;t)]\exp(i(\lambda^T - \mu^T)P). \tag{21}$$

The superscript T denotes transposition of the corresponding matrix. In (21), we have introduced the notation

$$\chi_1(\mu;t) = \frac{\partial}{\partial t}\ln h_1(\mu;t), \tag{22}$$

in which $h_1(\mu;t)$ is the one-dimensional characteristic function of the random process $\tilde{\Psi}(t)$.

To construct the equation for the two-dimensional characteristic function $g_2(\lambda_1,\lambda_2;t_1,t_2)$, we write down the expressions for the increments,

$$\tilde{\Psi}_1(t_1+\Delta t_1) = \Delta\tilde{\Psi}_1 + \tilde{\Psi}_1(t_1),$$
$$\tilde{\Psi}_2(t_2+\Delta t_2) = \Delta\tilde{\Psi}_2 + \tilde{\Psi}_2(t_2) \tag{23}$$

and the expression

$$\begin{aligned} &g_2(\lambda_1,\lambda_2;t_1+\Delta t_1,t_2+\Delta t_2) - g_2(\lambda_1,\lambda_2;t_1,t_2+\Delta t_2) \\ &\quad - g_2(\lambda_1,\lambda_2;t_1+\Delta t_1,t_2) + g_2(\lambda_1,\lambda_2;t_1,t_2) \\ &\quad = M[(\exp(i\lambda_1^T\Delta P_1)-1)(\exp(i\lambda_2^T\Delta P_2) \\ &\quad -1)\exp(i\lambda_1^T P_1 + i\lambda_2^T P_2)] \\ &\quad = M[(\exp(i\lambda_1^T b(P_1)\Delta\tilde{\Psi}_1)-1)(\exp(i\lambda_2^T b(P_2)\Delta\tilde{\Psi}_2) \\ &\quad -1)\exp(i\lambda_1^T P_1 + i\lambda_2^T P_2)], \end{aligned} \tag{24}$$

where $M[...]$ is the operation of finding the mathematical expectation. Here the subscripts 1 and 2 of $\tilde{\Psi}$, P, and λ indicate that they correspond to the different times t_1 and t_2.

We determine the expression for the function $M[\exp(i\mu_l^T\Delta\tilde{\Psi}_l)]$. For this, we calculate the mathematical expectation

$$\begin{aligned} &M[\exp(i\mu_l^T\tilde{\Psi}_l(t_l+\Delta t_l) + i\mu_j^T\tilde{\Psi}_j(t_j))] \\ &\quad = M[\exp(i\mu_l^T\Delta\tilde{\Psi}_l + i\mu_l^T\tilde{\Psi}_l(t_l) + i\mu_j^T\tilde{\Psi}_j(t_j))] \\ &\quad = M[\exp(i\mu_l^T\Delta\tilde{\Psi}_l)M[\exp(i\mu_l^T\tilde{\Psi}_l(t_l) \\ &\quad + i\mu_j^T\tilde{\Psi}_j(t_j))|\Delta\tilde{\Psi}_l]] \\ &\quad = M[\exp(i\mu_l^T\Delta\tilde{\Psi}_l)]M[\exp(i\mu_l^T\tilde{\Psi}_l(t_l) \\ &\quad + i\mu_j^T\tilde{\Psi}_j(t_j))], \\ &l=1,2, \quad j=1,2, \quad j=3-l, \end{aligned} \tag{25}$$

where $M[\ldots|\Delta\tilde{\Psi}_l]$ is the operation of finding the mathematical expectation when the increment $\Delta\tilde{\Psi}_1$ has a definite value, and μ_1 corresponds to the times t_l. It is assumed here that the process $\tilde{\Psi}(t)$ is completely described by the two-dimensional characteristic function $h_2(\mu_1,\mu_2;t_1,t_2)$. Therefore

$$\begin{aligned} &M[\exp(i\mu_l^T\tilde{\Psi}_l(t_l) + i\mu_j^T\tilde{\Psi}_j(t_j))|\Delta\tilde{\Psi}_l] \\ &\quad = M[\exp(i\mu_l^T\tilde{\Psi}_l(t_l) + i\mu_j^T\tilde{\Psi}_j(t_j))]. \end{aligned} \tag{26}$$

It follows from the expression (25) that

$$M[\exp(i\mu_l^T\Delta\tilde{\Psi}_l)] = \frac{h_2(\mu_j,\mu_l;t_j,t_l+\Delta t_l)}{h_2(\mu_1,\mu_2;t_1,t_2)},$$
$$l=1,2, \quad j=1,2, \quad j=3-l \tag{27}$$

or, to within terms of higher order in Δt_l,

$$M[\exp(i\mu_l^T\Delta\tilde{\Psi}_l)] = 1 + \chi_2^{(l)}(\mu_1,\mu_2;t_1,t_2)\Delta t_l, \tag{28}$$

where

$$\chi_2^{(l)}(\mu_1,\mu_2;t_1,t_2)=\frac{\partial}{\partial t_l}\ln h_2(\mu_1,\mu_2;t_1,t_2). \tag{29}$$

Next, we determine the function $M[\exp(i\mu_1^T\Delta\widetilde{\Psi}_1+i\mu_2^T\Delta\widetilde{\Psi}_2)]$. Carrying out manipulations analogous to (25), we obtain

$$M[\exp(i\mu_1^T\Delta\widetilde{\Psi}_1+i\mu_2^T\Delta\widetilde{\Psi}_2)]=\frac{h_2(\mu_1,\mu_2;t_1+\Delta t_1,t_2+\Delta t_2)}{h_2(\mu_1,\mu_2;t_1,t_2)} \tag{30}$$

or

$$M[\exp(i\mu_1^T\Delta\widetilde{\Psi}_1+i\mu_2^T\Delta\widetilde{\Psi}_2)]=(1+\chi_2^{(1)}(\mu_1,\mu_2;t_1,t_2)\Delta t_1)(1+\chi_2^{(2)}(\mu_1,\mu_2;t_1,t_2)\Delta t_2)+\Delta\chi_{20}(\mu_1,\mu_2;t_1,t_2)\Delta t_1\Delta t_2, \tag{31}$$

where

$$\chi_2^{(l)}(\mu_1,\mu_2;t_1,t_2)=\frac{\partial}{\partial t_l}\ln h_2(\mu_1,\mu_2;t_1,t_2),\quad l=1,2,$$

$$\Delta\chi_{20}(\mu_1,\mu_2;t_1,t_2)=\frac{\partial^2}{\partial t_1\partial t_2}\ln h_2(\mu_1,\mu_2;t_1,t_2). \tag{32}$$

After further manipulations with allowance for the expression (31) and retention of the terms of first order in the product $\Delta t_1\Delta t_2$, expression (24) reduces to

$$\frac{\partial^2 g_2(\lambda_1,\lambda_2;t_1,t_2)}{\partial t_1\partial t_2}=M[(\chi_2^{(1)}(b(P_1)^T\lambda_1,b(P_2)^T\lambda_2;t_1,t_2)\times\chi_2^{(2)}(b(P_1)^T\lambda_1,b(P_2)^T\lambda_2;t_1,t_2)+\chi_{20}(b(P_1)^T\lambda_1,b(P_2)^T\lambda_2;t_1,t_2))\times\exp(i\lambda_1^TP_1+i\lambda_2^TP_2)]. \tag{33}$$

To reduce Eq. (3) to the same form as was obtained in Ref. 7, we determine the difference

$$g_2(\lambda_j,\lambda_l;t_j,t_l+\Delta t_l)-g_2(\lambda_j,\lambda_l;t_j,t_l)=M[(\exp(i\lambda_l^T\Delta P_l)-1)\exp(i\lambda_1^TP_1+i\lambda_2^TP_2)]=M[(\exp(i\lambda_l^Tb(P_l)\Delta\widetilde{\Psi}_l)-1)\exp(i\lambda_1^TP_1+i\lambda_2^TP_2)],$$

$$l=1,2,\quad j=1,2,\quad j=3-l. \tag{34}$$

Manipulations similar to those above enable us to obtain the equation

$$\frac{\partial g_2(\lambda_j,\lambda_l;t_j,t_l)}{\partial t_l}=M[(\chi_2^{(l)}\times(b(P_1)^T\lambda_1,b(P_2)^T\lambda_2;t_1,t_2))\times\exp(i\lambda_1^TP_1+i\lambda_2^TP_2)],$$

$$l=1,2,\quad j=1,2,\quad j=3-l, \tag{35}$$

where the function $\chi_2^{(l)}$ is determined using the first expression in (32).

Equation (35) for $\lambda_j=0$ goes over to the equation for the one-dimensional characteristic function $g_1(\lambda_1;t_1)$ (Ref. 9):

$$\frac{\partial g_1(\lambda_l;t_l)}{\partial t_l}=M[\chi_1^{(l)}(b(P_l)^T\lambda_l;t_l)\exp(i\lambda_l^TP_l)],$$

$$l=1,2, \tag{36}$$

where we have introduced the notation

$$\chi_1^{(l)}(\mu_l;t_l)=\frac{\partial}{\partial t_l}\ln h_1(\mu_l;t_l),\quad l=1,2. \tag{37}$$

With allowance for the expressions (35)–(37), Eq. (33) can be represented in a form analogous to the one obtained in Ref. 7:

$$\left(\frac{\partial}{\partial t_1}-\hat{T}_1\right)\left(\frac{\partial}{\partial t_2}-\hat{T}_2\right)g_2=\Delta\hat{T}_2g_2, \tag{38}$$

where we have introduced the operators

$$\hat{T}_l=\frac{1}{2\pi}\int_{-\infty}^{\infty}dP_l\int_{-\infty}^{\infty}d\mu_l[\chi_1^{(l)}(b(P_l)^T\lambda_l;t_l)]\exp(i(\lambda_l^T-\mu_l^T)P_l),$$

$$l=1,2, \tag{39}$$

$$\Delta\hat{T}_2=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dP_1\int_{-\infty}^{\infty}d\mu_1\int_{-\infty}^{\infty}dP_2\int_{-\infty}^{\infty}d\mu_2\times[\Delta\chi_2(b(P_1)^T\lambda_1,b(P_2)^T\lambda_2;t_1,t_2)]\times\exp(i(\lambda_1^T-\mu_1^T)P_1+i(\lambda_2^T-\mu_2^T)P_2). \tag{40}$$

Here

$$\Delta\chi_2(\mu_1,\mu_2;t_1,t_2)=\frac{1}{h_2(\mu_1,\mu_2;t_1,t_2)}\left(\frac{\partial}{\partial t_1}-\chi_1^{(1)}\times(\mu_1;t_1)\right)\left(\frac{\partial}{\partial t_2}-\chi_1^{(2)}\times(\mu_2;t_2)\right)h_2(\mu_1,\mu_2;t_1,t_2). \tag{41}$$

The initial condition for Eq. (38) is

$$g_2(\lambda_1,\lambda_2;t_1,t_2)\Big|_{\substack{t_1=t_{10}\\t_2=t_{20}}}=g_{20}(\lambda_1,\lambda_2;t_{10},t_{20}). \tag{42}$$

With allowance for the form of the operators (39) and (40), Eq. (38) is the required equation for the two-dimensional characteristic function $g_2(\lambda_1,\lambda_2;t_1,t_2)$ of the non-Markovian process $P(t)$.

The above method also makes it possible to construct an equation for the n-dimensional characteristic function $g_n(\lambda_1,\dots,\lambda_n;t_1,\dots,t_n)$ in the form

$$\left(\frac{\partial}{\partial t_1}-\hat{T}_1\right)\cdots\left(\frac{\partial}{\partial t_n}-\hat{T}_n\right)g_n=\Delta\hat{T}_n g_n, \tag{43}$$

where the operator $\hat{T}_l$ is determined by means of the expression (39), and the operator $\Delta\hat{T}_n$ has the form

$$\Delta\hat{T}_n=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}dP_1\int_{-\infty}^{\infty}d\mu_1\dots\int_{-\infty}^{\infty}dP_n\int_{-\infty}^{\infty}d\mu_n \times[\Delta\chi_n(b(P_1)^T\lambda_1,\dots,b(P_n)^T\lambda_n;t_1,\dots,t_n)] \times\exp\left[i\sum_{k=1}^{n}(\lambda_k^T-\mu_k^T)P_k\right]. \tag{44}$$

Here

$$\Delta\chi_n(\mu_1,\dots,\mu_n;t_1,\dots,t_n)=\frac{1}{h_n(\mu_1,\dots,\mu_n;t_1,\dots,t_n)} \times\left(\frac{\partial}{\partial t_1}-\chi_1^{(1)}(\mu_1;t_1)\right)\dots\left(\frac{\partial}{\partial t_n}-\chi_1^{(n)}(\mu_n;t_n)\right)\times h_n(\mu_1,\dots,\mu_n;t_1,\dots,t_n). \tag{45}$$

The initial condition for Eq. (43) has the form

$$g_n(\lambda_1,\dots,\lambda_n;t_1,\dots,t_n)|_{t_l=t_{l0}}=g_{n0}(\lambda_1,\dots,\lambda_n;t_{10},\dots,t_{n0}),\quad l=1,\dots,n. \tag{46}$$

Equation (43) describes the non-Markovian random process $P(t)$ by means of the n-dimensional characteristic function $g_n(\lambda_1,\dots,\lambda_n;t_1,\dots,t_n)$.

With allowance for the form of the operators (21), (39), (40), and (44) and the expressions (37), (41), and (45), Eqs. (20), (38), and (43) enable us to construct a closed description of the Brownian motion in the non-Markovian case. For this, it is necessary to have *a priori* knowledge of the form of the n-dimensional characteristic function $h_n(\mu_1,\dots,\mu_n;t_1,\dots,t_n)$ of the external random process $\widetilde{\Psi}(t)$.

6. DESCRIPTION OF BROWNIAN MOTION BY MEANS OF A TWO-DIMENSIONAL CHARACTERISTIC FUNCTION

An equation for the two-dimensional characteristic function $g_2(\lambda_1,\lambda_2;t_1,t_2)$ of the random process $P(t)$ can be constructed if for the random process $\widetilde{\Psi}(t)$ we know its characteristic function $h_2(\mu_1,\mu_2;t_1,t_2)$, where $\mu_1=\{\eta_1,\kappa_1\}$ are the variables corresponding to the random processes $\widetilde{\theta}(t)$ and $\widetilde{W}(t)$. We represent the two-dimensional characteristic function $h_2(\eta_1,\kappa_1,\eta_2,\kappa_2;t_1,t_2)$ in a form that corresponds to the above requirement that the processes $P(t)$ and $\widetilde{\theta}(t)$ be independent:

$$h_2(\eta_1,\kappa_1,\eta_2,\kappa_2;t_1,t_2)=\exp\left[-\left(\kappa_1^2-\frac{i\eta_1}{mkT}\right)Dt_1-\left(\kappa_2^2-\frac{i\eta_2}{mkT}\right)Dt_2-2D\kappa_1\kappa_2\times\int_0^{t_1}d\tau_1\int_0^{t_2}d\tau_2\delta(\tau_2-\tau_1)+\left(\kappa_1^2-\frac{i\eta_1}{mkT}\right)\left(\kappa_2^2-\frac{i\eta_2}{mkT}\right)\times\int_0^{t_1}d\tau_1\int_0^{t_2}d\tau_2\Delta D(\tau_1,\tau_2)\right], \tag{47}$$

where $\Delta D(t_1,t_2)$ is a coefficient that describes the fluctuations of the coefficient of friction. In this case, the one-dimensional characteristic function of the random process $\widetilde{\Psi}(t)$ has the form

$$h_1(\eta,\kappa;t)=\exp\left[-\left(\kappa^2-\frac{i\eta}{mkT}\right)Dt\right]. \tag{48}$$

Then the equation for the two-dimensional characteristic function $g_2(\lambda_1,\lambda_2;t_1,t_2)$ of the random process $P(t)$ takes the form (38), and the corresponding equation for the one-dimensional characteristic function $g_1(\lambda;t)$ has the form (20).

By means of Eqs. (39) and (40), we can obtain expressions for the operators $\hat{T}_1$, 1=1,2 and $\Delta\hat{T}_2$ with allowance for the form of the characteristic function (47):

$$\hat{T}_l(\lambda_l;t_l)=-D\lambda_l\left(\lambda_l+\frac{1}{mkT}\frac{\partial}{\partial\lambda_l}\right),\quad l=1,2,$$

$$\Delta\hat{T}_2(\lambda_1,\lambda_2;t_1,t_2)=\delta\hat{T}_2(\lambda_1,\lambda_2;t_1,t_2)+\left(\int_0^{t_1}\delta\hat{T}_2(\lambda_1,\lambda_2;\tau_1,t_2)d\tau_1\right)\times\left(\int_0^{t_2}\delta\hat{T}_2(\lambda_1,\lambda_2;t_1,\tau_2)d\tau_2\right), \tag{49}$$

where

$$\delta\hat{T}_2(\lambda_1,\lambda_2;t_1,t_2)=-2D\delta(t_2-t_1)\lambda_1\lambda_2+\Delta D(t_1,t_2)\lambda_1\lambda_2\left(\lambda_1+\frac{1}{mkT}\frac{\partial}{\partial\lambda_1}\right)\times\left(\lambda_2+\frac{1}{mkT}\frac{\partial}{\partial\lambda_2}\right). \tag{50}$$

The expressions obtained above enable us to construct an equation for the two-dimensional distribution function $f_2(P_1,P_2;t_1,t_2)$, which takes the form

$$\left(\frac{\partial}{\partial t_1}-\hat{T}_1^{(f)}\right)\left(\frac{\partial}{\partial t_2}-\hat{T}_2^{(f)}\right)f_2=\Delta\hat{T}_2^{(f)}f_2, \tag{51}$$

where

$$\hat{T}_l^{(f)}(P_l;t_l)=D\frac{\partial}{\partial P_l}\left(\frac{\partial}{\partial P_l}+\frac{P_l}{mkT}\right),\quad l=1,2,$$

$$\Delta\hat{T}_2^{(f)}(P_1,P_2;t_1,t_2)=\delta\hat{T}_2^{(f)}(P_1,P_2;t_1,t_2)+\left(\int_0^{t_1}\delta\hat{T}_2^{(f)}(P_1,P_2;\tau_1,t_2)d\tau_1\right)\times\left(\int_0^{t_2}\delta\hat{T}_2^{(f)}(P_1,P_2;t_1,\tau_2)d\tau_2.\right. \quad (52)$$

Here we have introduced the notation

$$\delta\hat{T}_2^{(f)}(P_1,P_2;t_1,t_2)=2D\,\delta(t_2-t_1)\frac{\partial}{\partial P_1}\frac{\partial}{\partial P_2}+\Delta D(t_1,t_2)\frac{\partial}{\partial P_1}\frac{\partial}{\partial P_2}\times\left(\frac{\partial}{\partial P_1}+\frac{P_1}{mkT}\right)\left(\frac{\partial}{\partial P_2}+\frac{P_2}{mkT}\right). \quad (53)$$

The expressions (38) and (49)–(53) describe the Brownian motion when the effect of the particles of the medium on the Brownian particle can be completely described by a random process having a two-dimensional characteristic function of the form (47). The dependence of the momentum $P(t)$ of the Brownian particle on the time in this case is a non-Markov random process.

Note that the expression (51) is identical to the expression (7) obtained by the method of multidimensional distribution functions. At the same time, the operators (8) are a first approximation for the operators (52) and (53) obtained above.

7. CONCLUSIONS

Thus, the proposed method of describing the Brownian motion, which makes it possible to calculate the characteristics of the random process $P(t)$ with allowance for the deviation of the external disturbances from a Wiener process, can be regarded as one of the possible ways of constructing a non-Markovian theory of Brownian motion. The resulting expressions describe additional effects that can be detected by performing experiments to measure current fluctuations in small volumes of electrolytes.[14] The expressions (38) and (49)–(53) make it possible to take into account the real nature of the interaction of the particles of the medium with the Brownian particle, and this makes it possible to pose the problem of giving a more accurate description of the Brownian motion. The proposed method can be used to construct a non-Markov theory of kinetic processes with allowance for fluctuations of the transport coefficients.

The approach developed in this paper to a more accurate description of Brownian motion in real media presupposes the development of methods of solution of equations having the form (51). The examples of the description of diffusion in a medium with fluctuating transport coefficients and the solutions of equations of the type (51) considered in Ref. 7 are restricted to the case when the additional effects due to the fluctuations of the diffusion coefficient are sufficiently small. Then to solve Eq. (51) it is possible to use a perturbation method based in this case on the assumption of smallness of the right-hand side of Eq. (51). The development of general methods of solving equations of the form (51) is an important problem that requires additional investigations.

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