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APPLICATION OF INTEGRAL TRANSFORMS TO A DESCRIPTION OF THE BROWNIAN MOTION BY A NON-MARKOVIAN RANDOM PROCESS

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The one-dimensional Brownian motion and the Brownian motion of a spherical particle in an infinite medium are described by the conventional methods and integral transforms considering the entrainment of surrounding particles of the medium by the Brownian particle. It is demonstrated that fluctuations of the Brownian particle velocity represent a non-Markovian random process. A harmonic oscillator in a viscous medium is also considered within the framework of the examined model. It is demonstrated that for rheological models, random dynamic processes are also non-Markovian in character.

Key words: Brownian motion, oscillator, non-Markovian process, rheological models.

Application of the theory of Markovian processes to a description of the Brownian motion in actual physical media is approximate, because it ignores the special features of the interaction of the Brownian particles with particles of the medium [1, 2]. We note also that the processes proceeding in physical and technical systems are often non-Markovian ones [3–5]. In particular, the flicker noise observed in various physical processes can serve as an example of a non-Markovian process [6]. Various physical fluctuations of the kinetic coefficients observed experimentally (for example, fluctuations of the electrical conductivity) have the spectral density characteristic typical of the flicker noise. The flicker noise is the main type of noise limiting the sensitivity of electronic devices at low frequencies [7]. Actual radio engineering signals with amplitude and phase modulation by a combination of deterministic and random processes also belong to the non-Markovian process [5].

We note also that when a Markovian random process acts on a dynamic system, its response represents a non-Markovian random process. The sum of two Markovian processes is a non-Markovian process. The processes observed after integration of a Markovian process or finding of a sliding average of the process with independent values are also non-Markovian ones [4]. In particular, the coordinate of the Brownian particle calculated as an integral of its velocity is not always described by the model of a random Markovian process. The Wiener approximation is correct for the Brownian particles only for sufficiently long time intervals much longer than the particle relaxation time.

The above-indicated reasons demonstrate that we practically always use non-Markovian random processes to describe and to analyze actual physical processes and technical devices, and models of the Markovian process can be considered only as a first approximation.

BROWNIAN MOTION AS A MARKOVIAN PROCESS

To describe the Brownian motion, the approach based on the application of the stochastic Langevin equation is conventionally used [1, 8]. This approach allows the researchers to take advantage of the well-developed theory of stochastic differential systems [9, 10] which determines all necessary statistical characteristics of the velocity fluctuations of the Brownian particle motion.

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Let us consider motion of the spherical Brownian particle of radius R and mass M in the medium (liquid or gas) with kinematic viscosity ν and density ρ. The equation of motion for the particle has the form

$$M\frac{dV(t)}{dt} = F_0(t) + F_c(t) + \xi_V(t), \quad (1)$$

where $V(t)$ is the particle velocity, $F_c(t)$ is the resistance force, $F_0(t)$ is the sum of other external forces, and $\xi_V(t)$ is a random force. The resistance force $F_c(t) = F_{c0}(t)$ for the linear viscous medium is conventionally written as

$$F_{c0}(t) = -\gamma V(t), \quad (2)$$

where γ is the viscous friction coefficient which for the spherical particle can be written as $\gamma = 6\pi\rho\nu R$. For the spectral density of fluctuations of particle velocity $V(t)$, we can write [8]

$$G_V(\omega) = \frac{\alpha}{\omega^2 + \beta^2}, \quad (3)$$

where $\alpha = 2\gamma kT/M$, $\beta = \gamma/M$, k is the Boltzmann constant, T is the temperature of the viscous liquid in which the Brownian particle moves. For low frequencies, the spectral density $G_V(\omega)$ tends to a constant

$$G_V(\omega)\big|_{\omega\to 0} = \frac{2MkT}{\gamma}. \quad (4)$$

If the random process $\xi_V(t)$ represents a derivative of the process with independent increments, the Brownian particle velocity $V(t)$ is described by a Markovian random process, which allows any arbitrary L-dimensional characteristic functions and hence any multidimensional distribution functions to be assigned for it [9].

We note that the above-described approach can also be used to determine the temperature fluctuations of a body in thermal contact with a thermostat when its thermal flux is proportional to the difference between the body and thermostat temperatures.

ONE-DIMENSIONAL BROWNIAN MOTION

The simplest model of the Brownian motion considered above is applicable only when the viscous friction force acting in the medium can be described by Eq. (2). However, in the actual case, entrainment of particles of the medium surrounding the Brownian particle is observed in addition to its direct collisions [10]. This changes significantly the character of the viscous friction, and Eq. (2) is violated.

Consideration of the entrainment of the viscous liquid by a moving body calls for the application of integral equations for a description of motion; in turn, this calls for the application of the theory of non-Markovian processes [11]. We now demonstrate that even in the simplest case of one-dimensional Brownian motion in an infinite medium, it cannot be described by the Markovian process [12].

Let us consider the motion of a plane surface in a viscous liquid that occupies the half-space $x > 0$. We also consider that the plane is at the origin of coordinates (at $x = 0$) and moves with the velocity $V(t)$ in the direction perpendicular to the X axis lying in the plane (Fig. 1). The viscous friction force $F_c(t)$ acts on the plane from the medium together with the random force $\xi_V(t)$ (per unit area).

The plane motion in the viscous liquid is described by Eq. (1) (where the mass M per unit area is considered), and instead of Eq. (2), the formula

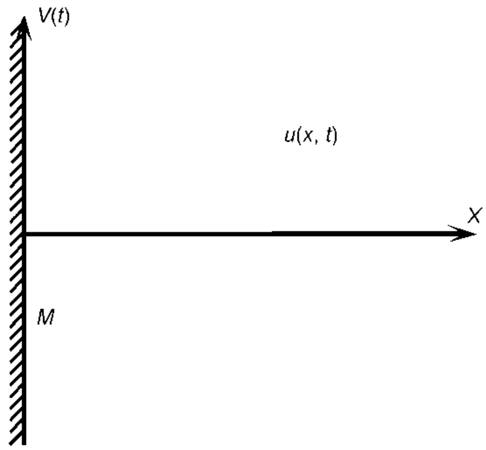

Fig. 1. Motion of a plane in a viscous liquid.

$$F_c(t) = \eta \left. \frac{\partial u(x,t)}{\partial x} \right|_{x=0} \tag{5}$$

should be written for the viscous friction force acting from the liquid, where $\eta = \nu\rho$ is the viscosity of the liquid, $u(x,t)$ is the velocity of the liquid parallel to the velocity of the plane $V(t)$ at each point. We consider that the deterministic force is equal to zero: $F_0(t) = 0$.

In the examined one-dimensional case, considering that the velocity of liquid is small, the equation for $u(x,t)$ at $x > 0$ assumes the form (see Section 24 of [10])

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{6}$$

with the following boundary and initial conditions:

$$u(x,0) = 0\,, \tag{7}$$

$$u(0,t) = V(t)\,. \tag{8}$$

Then a solution of Eq. (6) with allowance for conditions (7) and (8) assumes the form [13]

$$u(x,t) = \frac{1}{2\sqrt{\pi\nu}} \int_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4\nu(t-\tau)} \right] V(\tau) d\tau\,. \tag{9}$$

Let us find a derivative of Eq. (9) with respect to the coordinate x:

$$\frac{\partial u(x,t)}{\partial x} = \frac{1}{2\sqrt{\pi\nu}} \int_0^t \left[\frac{1}{(t-\tau)^{3/2}} - \frac{2x^2}{4\nu(t-\tau)^{5/2}} \right] \exp\left[-\frac{x^2}{4\nu(t-\tau)} \right] V(\tau) d\tau\,. \tag{10}$$

Integrating Eq. (10) by parts, we obtain

$$\frac{\partial u(x,t)}{\partial x} = -\frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{1}{\sqrt{t-\tau}}\exp\left[-\frac{x^2}{4\nu(t-\tau)}\right]\frac{dV(\tau)}{d\tau}d\tau. \tag{11}$$

After substitution of Eq. (11) into Eq. (5), we obtain the dependence of the force $F_c(t)$ on the velocity $V(t)$:

$$F_c(t) = -\frac{\eta}{\sqrt{\pi\nu}}\int_0^t \frac{1}{\sqrt{t-\tau}}\frac{dV(\tau)}{d\tau}d\tau. \tag{12}$$

In the examined case, this formula replaces Eq. (2). Formula (12) was derived by other methods in [10] (see Section 24).

Thus, the description of the velocity fluctuations of the plane surface in the viscous liquid filling the half-space is reduced to a solution of system of equations (1) and (12). Since equation (12) is integral, the random processes $F_c(t)$ and $V(t)$ are non-Markovian ones.

We note that the description of the thermal conductivity in a one-dimensional infinite medium is reduced to expression analogous to Eq. (12) [12]. In a study of the internal friction caused by wave scattering on random inhomogeneities of the medium in one-dimensional elastic systems, expressions analogous to Eq. (12) arise (see [14, 15]).

System of equations (1) and (12) can be written down in the form of the second-order Volterra integral equation [16]

$$Z(t) + A\int_0^t Z(\tau)\frac{d\tau}{\sqrt{t-\tau}} = \xi(t), \tag{13}$$

where

$$Z(t) = \frac{dV(t)}{dt},\ A = \frac{\eta}{M\sqrt{\pi\nu}},\ \xi(t) = \frac{\xi_V(t)}{M}. \tag{14}$$

It is obvious that $Z(t)$, being a solution of integral equation (13), is a non-Markovian random process.

A solution of integral equation (13) has the form

$$Z(t) = \int_0^t \left(\delta(t-\tau) - R(t,\tau)\right)\xi(\tau)d\tau, \tag{15}$$

where the resolvent is (see Section 3 of Chapter VIII in [21])

$$R(t,\tau) = \frac{1}{t-\tau}\sum_{k=1}^{\infty}(-1)^{k+1} r_k (t-\tau)^{\frac{k}{2}}, \quad r_k = \frac{A^k \pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}. \tag{16}$$

Here $\Gamma(x)$ is the gamma-function. Figure 2 shows the calculated function $R(t-\tau)$. It can be seen that the function $R(t-\tau)$ sharply decreases with increasing difference $t-\tau$.

Taking advantage of the method of describing non-Markovian random processes presented in [11], for one- and L-dimensional characteristic functions of the random process $Z(t)$ specified by linear integral expression (15), we obtain

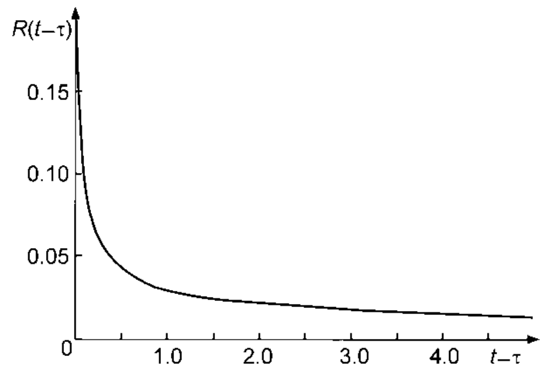

Fig. 2. Plot of the function $R(t-\tau)$.

$$g_1(\lambda;t)=\exp\left[-\frac{1}{2}\sigma\lambda^2\left(r_1^2\ln\frac{t}{\delta t}+\sum_{k=2}^{\infty}r_k^2\frac{t^{k-1}}{k-1}+4\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}(-1)^{m+n}r_m r_n\frac{t^{\frac{m+n-2}{2}}}{m+n-2}+r_2+\delta(t)\right)\right], \tag{17}$$

$$g_L(\lambda_1,...,\lambda_L;t_1,...,t_L)=\exp\left[-\frac{1}{2}\sigma\left(\sum_{l,k=1}^{L}\lambda_l\lambda_k\left(\delta(t_l-t_k)+R(t_l,t_k)+\sum_{m,n=1}^{\infty}r_m r_n f_{mn}\right)\right)\right], \tag{18}$$

where σ is the intensity of the random process $\xi(t)$ whose distribution is described by a Gaussian curve, δt is a small positive value, $\delta(x)$ is the delta-function,

$$f_{mn}=\begin{cases} 2\ln\dfrac{\sqrt{t_l}+\sqrt{t_k}}{\sqrt{t_l-t_k+\delta t}+\sqrt{\delta t}} & \text{for } m,n=1,\\[2ex] \displaystyle\sum_{s=0}^{\frac{m}{2}-1}C_s^{\frac{m}{2}-1}(-1)^{\frac{m}{2}-s-1}\frac{t_l^{s+\frac{n}{2}}-(t_l-t_k)^{\frac{m+n}{2}-1}}{\frac{n}{2}+s} & \text{for even } m,\\[2ex] \displaystyle\sum_{s=1}^{\frac{n}{2}-1}C_s^{\frac{n}{2}-1}\frac{(t_l-t_k)^{\frac{n}{2}-s-1}\,t_k^{\frac{m}{2}+s}}{\frac{m}{2}+s} & \text{for odd } m \text{ and even } n,\\[2ex] \displaystyle t_l^{a+\frac{3}{2}}\sum_{s=1}^{a}\frac{(2a+1)(2a-1)...(2a-2s+3)(t_k-t_l)^s}{(a+b+2)(a+b+1)...(a+b-s+2)}t_k^{a-s+\frac{1}{2}} \\ \displaystyle-\sum_{s=0}^{b}\frac{(2a+1)(2a-1)...(2a-2s+3)}{2^s(a+b+2)(a+b+1)...(b-s+1)}\frac{(t_k-t_l)^{a+s+1}a!!\sqrt{t_k}}{2^{a+1}} \\ \displaystyle+\frac{a!!b!!(t_k-t_l)^{a+b+2}}{(a+b+2)!2^{a+b+2}(-1)^{b+1}}2\ln\frac{\sqrt{t_k}+\sqrt{t_l}}{\sqrt{t_l-t_k+\delta t}+\sqrt{\delta t}} & \text{for other values of } m \text{ and } n. \end{cases} \tag{19}$$

Here C_a^b is the corresponding binomial coefficient, $a=\dfrac{m-3}{2}$, and $b=\dfrac{n-3}{2}$.

Expressions (17) and (18) so obtained allow any arbitrary characteristics of the random process $Z(t)$ to be calculated. In particular, for the correlation function $\langle Z(t_1)Z(t_2)\rangle$ we obtain

$$\langle Z(t_1)Z(t_2)\rangle = \sigma\left(\delta(t_2-t_1)+R(t_2,t_1)+\sum_{m,n=1}^{\infty} r_m r_n f_{mn}\right), \tag{20}$$

where the function f_{mn} is given by Eq. (19) in which substitutions $t_k = t_1$, $t_l = t_2$ have been made. Equation (20) allows the spectral density of the random process $Z(t)$ to be calculated according to the definition [9]

$$G_Z(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \langle Z(t)Z(t-\tau)\rangle e^{-i\omega\tau} d\tau . \tag{21}$$

Taking the Laplace transform of Eq. (13), we obtain

$$\hat{Z}(p) = \frac{\sqrt{p}}{\sqrt{p}+A\sqrt{\pi}}\hat{\xi}(p), \tag{22}$$

where $\hat{Z}(p)$ and $\hat{\xi}(p)$ are the Laplace transforms of functions $Z(t)$ and $\xi(t)$, respectively.

Since the spectral density of the process $\xi(t)$ is constant and equal to its intensity

$$G_\xi = \sigma, \tag{23}$$

from formula (22) at $t \to \infty$ we obtain the spectral density of the process $Z(t)$:

$$G_Z(\omega) = \left|\frac{\sqrt{i\omega}}{\sqrt{i\omega}+A\sqrt{\pi}}\right|^2 \sigma \tag{24}$$

or

$$G_Z(\omega) = \frac{\omega\sigma}{\omega + A\sqrt{2\pi\omega}+\pi A^2}. \tag{25}$$

Figure 3 compares plots of the spectral density for $A = 0.1\ \mathrm{s}^{-1/2}$ calculated from formulas (20) and (21) (for $t = 10^5$ s) and from formula (25). Good agreement of the results obtained by different methods can be seen. The difference at high frequencies is most likely caused by the limited number of terms of infinite series (16) considered in numerical calculations.

Equation (25) with allowance for Eq. (14) allows us to calculate the spectral density of fluctuations of the velocity $V(t)$:

$$G_V(\omega) = \frac{\sigma}{\omega\left(\omega + A\sqrt{2\pi\omega}+\pi A^2\right)}, \tag{26}$$

where

$$\sigma = \frac{2\gamma kT}{M^2}. \tag{27}$$

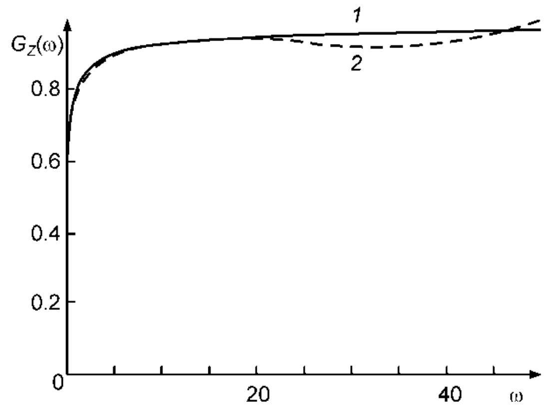

Fig. 3

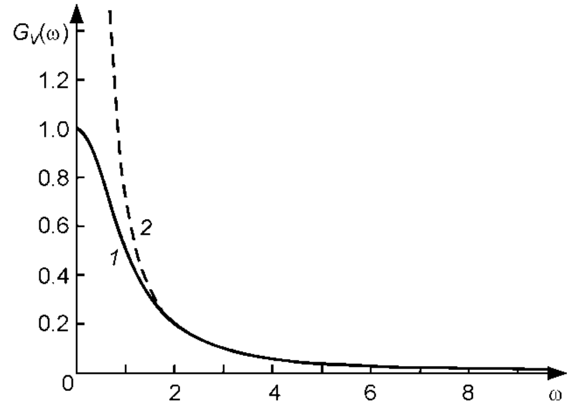

Fig. 4

Fig. 3. Plots of the spectral density $G_Z(\omega)$ calculated by analytical formula (25) (curve *1*) and numerically (curve *2*).

Fig. 4. Plots of the spectral density $G_V(\omega)$ calculated by formulas (3) (curve *1*) and (26) (curve *2*).

For low frequencies $\omega << A^2$, Eq. (26) assumes the form

$$G_V(\omega)\big|_{\omega<<A^2} = \frac{\sigma}{\pi A^2 \omega}, \tag{28}$$

or, with allowance for Eqs. (14) and (27),

$$G_V(\omega)\big|_{\omega<<A^2} = \frac{2\gamma kT}{\eta\rho\omega}. \tag{29}$$

From Eq. (29) it follows that fluctuations of the plane surface velocity $V(t)$ in the viscous liquid represent flicker noise [6], which at low frequencies is inversely proportional to the frequency.

Figure 4 shows the dependences of the spectral density $G_V(\omega)$ calculated by Eqs. (3) and (26). From the figure it can be seen that at high frequencies, these two dependences are similar, whereas at low frequencies, they differ significantly due to the presence of the flicker noise in the case described by Eq. (26).

We note that temperature fluctuations of the plane surface in the problem of one-dimensional thermal conductivity also have the spectral density described by Eq. (26) and hence, the flicker noise is also typical of them. With allowance for the temperature dependence of the kinetic coefficients, this leads to the fluctuations of these coefficients with the spectral density of the flicker noise at low frequencies.

Thus, the simple model problem of one-dimensional Brownian motion demonstrates that the fluctuations of the velocity $V(t)$ represent the non-Markovian random process whose special feature is the presence of flicker noise at low frequencies.

BROWNIAN MOTION OF A SPHERICAL PARTICLE IN AN INFINITE MEDIUM IN THE PRESENCE OF A RESTORING FORCE

To describe the Brownian motion of a spherical particle with allowance for the entrainment of particles of the medium, expression (see problem 7 in Section 24 of [10])

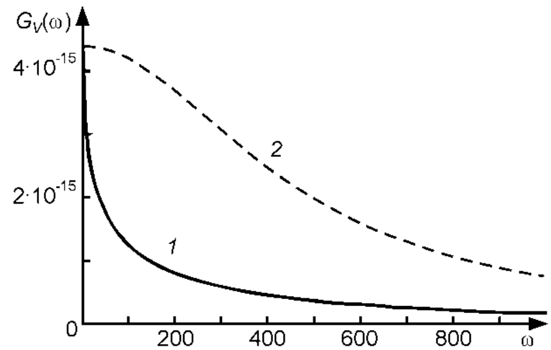

Fig. 5. Plots of the spectral densities calculated by Eqs. (3) (curve *2*) and (32) (curve *1*) for ρ = 1000 kg/m^3, ν = 10^{-6} m^2/s, R = 10^{-4} m, M = 4·10^{-9} kg, and T = 300 K.

$$F_c(t) = -2\pi\rho R^3\left[\frac{1}{3}\frac{dV(t)}{dt} + \frac{3\nu}{R^2}V(t) + \frac{3}{R}\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{dV(\tau)}{d\tau}\frac{d\tau}{\sqrt{t-\tau}}\right] \tag{30}$$

should be used instead of Eq. (2). The particle motion in Eq. (30) started at time $t = 0$. Formula (30) was derived by finding the resistance force acting on the spherical particle executing harmonic oscillations in the viscous liquid (for small Reynolds numbers) with subsequent use of this solution for finding of the resistance force acting on the particle moving arbitrarily with the velocity $V(t)$ that can be represented by the Fourier integral.

Substituting Eq. (30) into Eq. (1), we derive the equation of Brownian motion of the spherical particle:

$$\left(M + \frac{2}{3}\pi\rho R^3\right)\frac{dV(t)}{dt} + 6\pi\rho\nu RV(t) + 6\rho R^2\sqrt{\pi\nu}\int_0^t \frac{dV(\tau)}{d\tau}\frac{d\tau}{\sqrt{t-\tau}} = F_0(t) + \xi_V(t). \tag{31}$$

In [17] it was demonstrated that in this case, the velocity fluctuations and the coordinates of the Brownian particle represented non-Markovian random processes, and their statistical characteristics differed from those of the classical Brownian particles. In particular, expression

$$G_V(\omega) = \frac{G_\xi}{\omega^2 + AB\sqrt{2\pi\omega^3} + A^2B^2\pi\omega + A^2B\sqrt{2\pi\omega} + A^2}, \tag{32}$$

was obtained in [17] for the spectral density of velocity fluctuations, where

$$A = \frac{6\pi\rho\nu R}{M + \frac{2}{3}\pi\rho R^3}, \quad B = R\sqrt{\frac{1}{\pi\nu}}, \tag{33}$$

$$G_\xi = \frac{12\pi\rho\nu RkT}{\left(M + \frac{2}{3}\pi\rho R^3\right)^2}. \tag{34}$$

A comparison of formulas (3) and (32) demonstrates that the use of Eq. (30) instead of Eq. (2) for the resistance force leads to essential difference between the spectral densities of particle velocity fluctuations, especially at low and intermediate frequencies. Figure 5 compares plots of spectral densities given by formulas (3) and (32).

We now consider a restoring force acting on the spherical particle along the X axis in addition to the random and resistance forces. In this case, Eq. (31) assumes the form

$$Z(t)+A\int_0^t\left(1+\frac{B}{\sqrt{t-\tau}}\right)Z(\tau)d\tau=-\tilde{k}X(t)+\tilde{\xi}(t), \tag{35}$$

where

$$Z(t)=\frac{dV(t)}{dt},\ \tilde{k}=\frac{k}{M+\frac{2}{3}\pi\rho R^3},\ \tilde{\xi}(t)=\frac{\xi_V(t)}{M+\frac{2}{3}\pi\rho R^3}. \tag{36}$$

Applying the procedure used in the derivation of formula (26), we obtain for spectral densities of processes $Z(t)$ and $V(t)$:

$$G_Z(\omega)=\frac{\omega^4}{\omega^4+AB\sqrt{2\pi\omega^7}+A^2B^2\pi\omega^3+A^2B\sqrt{2\pi\omega^5}+(A^2-2\tilde{k})\omega^2-\tilde{k}AB\sqrt{2\pi\omega^3}+\tilde{k}^2}G_{\xi}, \tag{37}$$

$$G_V(\omega)=\frac{\omega^2}{\omega^4+AB\sqrt{2\pi\omega^7}+A^2B^2\pi\omega^3+A^2B\sqrt{2\pi\omega^5}+(A^2-2\tilde{k})\omega^2-\tilde{k}AB\sqrt{2\pi\omega^3}+\tilde{k}^2}G_{\xi}. \tag{38}$$

The spectral density for coordinate $X(t)$ can be easily found from Eq. (38). We obtain

$$G_X(\omega)=\frac{1}{\omega^4+AB\sqrt{2\pi\omega^7}+A^2B^2\pi\omega^3+A^2B\sqrt{2\pi\omega^5}+(A^2-2\tilde{k})\omega^2-\tilde{k}AB\sqrt{2\pi\omega^3}+\tilde{k}^2}G_{\xi}. \tag{39}$$

Let us compare the last expression with the spectral density for the classical oscillator obtained from Eq. (39) for $B=0\ \mathrm{s}^{1/2}$ when terms comprising $\frac{2}{3}\pi\rho R^3$ in the denominators of Eqs. (33) and (34) are set equal to zero. In this case, Eq. (39) is reduced to the formula

$$G_X^{\mathrm{cl}}(\omega)=\frac{1}{\omega^4+(A'^2-2k')\omega^2+k'^2}G'_{\xi}, \tag{40}$$

where $A'=\frac{6\pi\rho\nu R}{M}$, $k'=\frac{k}{M}$, and the spectral noise density is $G'_{\xi}=\frac{12\pi\rho\nu RkT}{M^2}$.

Figures 6 and 7 show the spectral densities calculated from Eqs. (39) and (40) for the following values of the parameters: $\nu=10^{-6}\ \mathrm{m^2/s}$, $\rho=10^3\ \mathrm{kg/m^3}$, $k=10^{-2}\ \mathrm{N/m}$, and R = 10 (Fig. 6) and 100 μm (Fig. 7). The particle density was set equal to the density of the medium.

From the plots it can be seen that the curves become similar with increasing particle sizes, and higher amplitudes of the spectral density correspond to the classical case. This effect becomes most pronounced for large particles. From the plots it can also be seen that the maximum of the spectral density in the classical case is observed at higher resonant frequencies. This is a consequence of the fact that in the nonclassical description, a certain *effective* mass $\frac{2}{3}\pi\rho R$ is added to the particle mass.

Thus, the description of the Brownian spherical particle motion in the infinite viscous medium has allowed us to establish that its velocity fluctuations represent the non-Markovian random process. The resonant curves drawn for

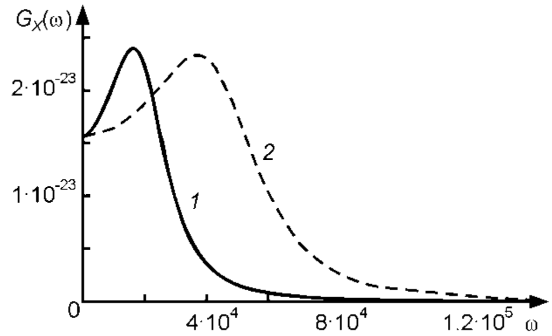

Fig. 6

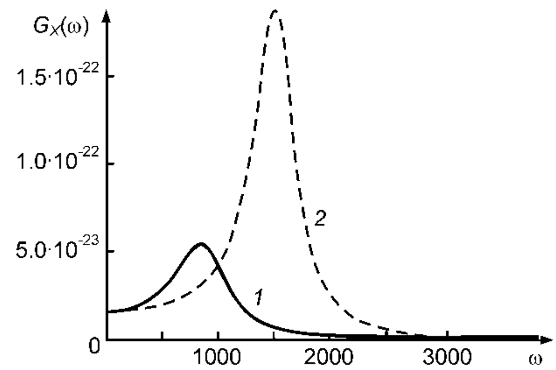

Fig. 7

Fig. 6. Plots of the spectral densities calculated from Eqs. (39) (curve *1*) and (40) (curve *2*) for $\nu = 10^{-6}$ m^2/s, $\rho = 10^3$ kg/m^3, $k = 10^{-2}$ N/m, and $R = 10$ μm.

Fig. 7. Plots of the spectral densities calculated from Eqs. (39) (curve *1*) and (40) (curve *2*) for $\nu = 10^{-6}$ m^2/s, $\rho = 10^3$ kg/m^3, $k = 10^{-2}$ N/m, and $R = 100$ μm.

the mechanical oscillator put in the viscous infinite medium differ from the classical ones. The last result can be important for devices of fluctuation damping.

RHEOLOGICAL MODELS

When considering physical processes in rheological media, relationships of their quantities are conventionally described by the corresponding differential equations. However, when time scales in the rheological medium are smaller or comparable with relaxation times in the medium, this description appears too rough [18]. This is the case, in particular, for the Brownian particle moving in the rheological medium or for short-period external forces acting on the medium. One of the approaches used for these problems is a description of relationships among the corresponding physical quantities by integral operators. For example, the relationship between the stress $\sigma(t)$ and strain $\varepsilon(t)$ of a rheological material subject to linear deformation can be expressed by the integral Volterra operators of the second kind [16]:

$$\varepsilon(t) = \frac{1}{E}\left(\sigma(t) + \int_0^t K(t-\tau)\sigma(\tau)\,d\tau\right), \tag{41}$$

$$\sigma(t) = E\left(\varepsilon(t) - \int_0^t R(t-\tau)\varepsilon(\tau)\,d\tau\right). \tag{42}$$

Here $K(t-\tau)$ is the creep kernel, $R(t-\tau)$ is the relaxation kernel, and E is the instantaneous elasticity modulus. In Eqs. (41) and (42), it is assumed that stresses and strains are absent when $t < 0$. Equation (42) is a solution of Eq. (41); because of this, the creep and relaxation kernels $K(t-\tau)$ and $R(t-\tau)$ are interrelated [16]. The concrete behavior of the creep and relaxation kernels is determined experimentally. Experimental dependences of the creep kernel are most often approximated by the expression of the form [19]

$$K(t-\tau)=\frac{A}{(t-\tau)^{\alpha}}, \tag{43}$$

where A and α are constants and $0<\alpha\le 1$. Such kernels are used, for example, to describe melting of metals, polymer solutions, etc.

As demonstrated in [20], the integral equation of motion for a system with one degree of freedom under the action of variable force $f(t)$ in the medium that obeys the law of inherited viscoelasticity has the form

$$\ddot{q}(t)=-kq(t)+\int_0^t \Phi(t-\tau)\varphi(q(\tau),\tau)\,d\tau+f(t), \tag{44}$$

where q is the generalized coordinate, k is the elasticity coefficient, $\Phi(t-\tau)$ is the kernel describing the influence of the inherited viscoelasticity, and $\varphi(y,\tau)$ describes the dependence of the force on the generalized coordinate, generally nonlinear one.

The application of the last formula to the Brownian particle motion (in this case, $X(t)$ plays the role of the generalized coordinate) with a power-law kernel of influence similar to Eq. (43) is reduced to the problem considered above. Thus, the Brownian particle motion in rheological media is described by integral operators and by analogy with the foregoing, represents a non-Markovian process.

The above description of the one-dimensional Brownian motion and Brownian motion of the spherical particle in the infinite viscous medium by random non-Markovian processes demonstrates their significant difference from the processes investigated by the classical methods. The results obtained can be important for consideration of various random processes for which the model of Brownian motion is applicable.

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